

The symplectic structure on the moduli space of line bundles on a noncommutative Azumaya surface

Fabian Reede*

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Abstract

In this note we prove that the moduli space of torsion-free modules of rank one over an Azumaya algebra on a $K3$ -surface is an irreducible symplectic variety deformation equivalent to a Hilbert scheme of points on the $K3$ -surface.

Introduction

Assume X is a smooth projective $K3$ -surface over \mathbb{C} and let \mathcal{A} be an Azumaya algebra on X . The algebra \mathcal{A} is an example of a so called noncommutative projective surface and also an example of what is called a Calabi-Yau order, that is a noncommutative analogue of a classical $K3$ -surface.

Locally projective \mathcal{A} -modules of rank one can be considered as line bundles on this noncommutative surface. In ([HS05]) the authors construct moduli schemes for such line bundles. These schemes can be seen as noncommutative versions of the usual Picard schemes. By allowing torsion-free \mathcal{A} -modules and by fixing invariants, for example the Mukai vector, these moduli schemes are shown to be projective schemes over \mathbb{C} . Furthermore the authors show that these moduli schemes are smooth and possess a symplectic structure. So one can ask the question: are these moduli schemes irreducible symplectic varieties (hyperkähler manifolds)?

There are 4 known classes of hyperkähler manifolds: we have the Hilbert schemes of points on a smooth projective $K3$ -surface and the generalized Kummer varieties associated to an abelian surface, both of these classes are due to Beauville. Furthermore there is a class of 6-dimensional examples and one class of 10-dimensional examples, both classes are due to O'Grady using symplectic desingularization. All other known examples of hyperkähler manifolds are deformation equivalent to one of the four examples mentioned above. So if the moduli spaces of line bundles on a noncommutative Azumaya surface are hyperkähler manifolds, do they give rise to a new class or are they also deformation equivalent to one of the four known classes?

In this note we prove that these moduli spaces are in fact hyperkähler manifolds and that they do not give rise to a new class of hyperkähler manifolds. We prove that they belong to the first class by showing that they are deformation equivalent to appropriate Hilbert schemes. This is done by comparing the moduli spaces to moduli spaces constructed by Yoshioka in ([Yos06]). Yoshioka proves that they are hyperkähler manifolds deformation equivalent to Hilbert schemes, hence so are the moduli spaces of line bundles on a noncommutative Azumaya surface.

*Mary Immaculate College, Limerick, fabianreede@gmx.net

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1 Modules over Azumaya algebras and Brauer-Severi varieties

In this section we assume X is any fixed scheme of finite type over \mathbb{C} , that is $X \in Sch_{\mathbb{C}}$. Following ([HL10]) we denote by $Sch_{\mathbb{C}}$ the category of schemes of finite type over \mathbb{C} and for $X \in Sch_{\mathbb{C}}$ we denote the category of schemes of finite type over X by Sch_X .

Definition 1.1. *A sheaf \mathcal{A} of \mathcal{O}_X -algebras is called Azumaya algebra if for every closed point $x \in X$ the fiber $\mathcal{A}(x) = \mathcal{A} \otimes k(x)$ is a central simple algebra over the residue field $k(x)$.*

Remark 1.2. The rank of an Azumaya algebra \mathcal{A} is always a square, that is we have $rk(\mathcal{A}) = r^2$. We also note that for every morphism $f : T \rightarrow X$ the \mathcal{O}_T -algebra $f^*\mathcal{A}$ is an Azumaya algebra on T .

Define the functor

$$BS(\mathcal{A}) : (Sch_X)^{op} \rightarrow Sets$$

which sends an X -scheme $f : T \rightarrow X$ to the set of left ideals $\mathcal{I} \subseteq \mathcal{A}_T$ such that $\mathcal{A}_T/\mathcal{I}$ is a locally free \mathcal{O}_T -module of rank $r(r-1)$, here $\mathcal{A}_T = f^*\mathcal{A}$.

Lemma 1.3. *The functor $BS(\mathcal{A})$ is representable by an X -scheme $\pi : BS(\mathcal{A}) \rightarrow X$. The morphism π is faithfully flat and proper and exhibits $BS(\mathcal{A})$ as an étale \mathbb{P}^{r-1} -bundle over X .*

Proof. For the first assertion see ([GW10, 8.13, Exercise 8.14.]) and for the second assertion see ([Gro68, Théorème 8.2, Corollaire 8.3]). \square

Remark 1.4. For every X -scheme $f : T \rightarrow X$ we get by functoriality a canonical isomorphism

$$BS(\mathcal{A}_T) = BS(f^*\mathcal{A}) \cong BS(\mathcal{A}) \times_X T.$$

The scheme $BS(\mathcal{A})$ is called the Brauer-Severi variety associated to \mathcal{A} . In the following we will just write $\pi : Y \rightarrow X$ for this X -scheme.

There is a canonically defined locally free sheaf G of rank r on Y , which is unique up to scalars ([Yos06, Lemma 1.1.]), which sits in the Euler sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow G \longrightarrow \mathcal{T}_{Y/X} \longrightarrow 0.$$

Using ideas of Yoshioka, see ([Yos06, Lemma 1.5.]), we define the following subcategory of $Coh(Y)$:

Definition 1.5. Assume $\pi : Y \rightarrow X$ is a Brauer-Severi variety, then we define:

$$\text{Coh}(Y, X) := \{E \in \text{Coh}(Y) \mid \pi^* \pi_*(E \otimes G^\vee) \xrightarrow{\sim} E \otimes G^\vee\}.$$

Here the morphism is the canonical morphism coming from the adjunction between π^* and π_* .

Furthermore we denote the category of coherent sheaves on X having the structure of a left respectively right \mathcal{A} -module by $\text{Coh}_l(X, \mathcal{A})$ respectively $\text{Coh}_r(X, \mathcal{A})$.

Lemma 1.6. Assume $\pi : Y \rightarrow X$ is a Brauer-Severi variety, then for $E \in \text{Coh}(X)$ we have

$$R^i \pi_*(\pi^* E) \cong \begin{cases} E & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Proof. For every $i \geq 0$ there is a natural morphism $R^i \pi_*(\mathcal{O}_Y) \otimes E \rightarrow R^i \pi_*(\pi^* E)$. Since Y is an étale \mathbb{P}^{r-1} -bundle over X we have

$$R^i \pi_* \mathcal{O}_Y \cong \begin{cases} \mathcal{O}_X & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

It is therefore enough to prove this after a faithfully flat étale base change $f : X' \rightarrow X$ that splits \mathcal{A} .

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ \pi' \downarrow & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

So we have $f^* \mathcal{A} \cong \text{End}_{\mathcal{O}_{X'}}(\mathcal{E})$ for a locally free sheaf \mathcal{E} on X' . But then by (1.4)

$$Y' = Y \times_X X' \cong BS(f^* \mathcal{A})$$

and using Morita equivalence we see that $BS(f^* \mathcal{A}) = BS(\text{End}_{\mathcal{O}_{X'}}(\mathcal{E})) \cong \mathbb{P}(\mathcal{E}^\vee)$. Now the result follows from ([TT90, 4.5.(f)]). \square

Remark 1.7. The Brauer-Severi variety $\pi : Y \rightarrow X$ associated to an Azumaya algebra \mathcal{A} has the following splitting property: there is an isomorphism of \mathcal{O}_Y -algebras on Y

$$\pi^* \mathcal{A} \cong \text{End}_{\mathcal{O}_Y}(G)^{op},$$

see ([Qui73, 8.4.]). This shows that we have $\pi^*(\mathcal{A}^{op}) \cong \text{End}_{\mathcal{O}_Y}(G)$ and hence

$$\mathcal{A}^{op} \cong \pi_* \pi^*(\mathcal{A}^{op}) \cong \pi_*(\text{End}_{\mathcal{O}_Y}(G)).$$

Remark 1.8. Using Morita equivalence one can see that there is an equivalence of categories

$$\text{Coh}_r(Y, \text{End}_{\mathcal{O}_Y}(G)) \cong \text{Coh}(Y).$$

The equivalence is given by $E \mapsto E \otimes_{\text{End}_{\mathcal{O}_Y}(G)} G$ with inverse $F \mapsto F \otimes G^\vee$, see for example ([GW10, 8.12.]).

Lemma 1.9. Assume \mathcal{A} is an Azumaya algebra on X and let $\pi : Y \rightarrow X$ be the associated Brauer-Severi variety. Then there is an equivalence of categories:

$$\text{Coh}_l(X, \mathcal{A}) \cong \text{Coh}(Y, X).$$

Proof. The category $Coh_l(X, \mathcal{A})$ is isomorphic to $Coh_r(X, \mathcal{A}^{op})$, hence it is enough to show that $Coh_r(X, \mathcal{A}^{op})$ and $Coh(Y, X)$ are equivalent. For this we define the following functors:

$$\begin{aligned} F : Coh_r(X, \mathcal{A}^{op}) &\rightarrow Coh(Y, X), E \mapsto \pi^* E \otimes_{\pi^*(\mathcal{A}^{op})} G \\ H : Coh(Y, X) &\rightarrow Coh_r(X, \mathcal{A}^{op}), E \mapsto \pi_*(E \otimes G^\vee) \end{aligned}$$

First of all we need to verify that F and H are well-defined. That is we need to see that

- i) $\pi^* E \otimes_{\pi^*(\mathcal{A}^{op})} G \in Coh(Y, X)$
- ii) $\pi_*(E \otimes G^\vee) \in Coh_r(X, \mathcal{A}^{op})$.

For i) we look at the canonical morphism

$$\pi^* \pi_*((\pi^* E \otimes_{\pi^*(\mathcal{A}^{op})} G) \otimes G^\vee) \rightarrow (\pi^* E \otimes_{\pi^*(\mathcal{A}^{op})} G) \otimes G^\vee.$$

But by definition we have $G \otimes G^\vee \cong \mathcal{E}nd_{\mathcal{O}_Y}(G) \cong \pi^*(\mathcal{A}^{op})$. So we have to see that $\pi^* \pi_* \pi^* E \rightarrow \pi^* E$ is an isomorphism. But this follows since $E \rightarrow \pi_* \pi^* E$ is an isomorphism by (1.6) and (π^*, π_*) is a pair of adjoint functors.

For ii) we just note that G^\vee is a right $\mathcal{E}nd_{\mathcal{O}_Y}(G)$ -module, hence so is $E \otimes G^\vee$. Then $\pi_*(E \otimes G^\vee)$ is a right $\pi_*(\mathcal{E}nd_{\mathcal{O}_Y}(G)) \cong \mathcal{A}^{op}$ -module by (1.7).

Now we study $F \circ H$ and $H \circ F$: for $E \in Coh_r(X, \mathcal{A}^{op})$ we have:

$$H(F(E)) = \pi_*((\pi^* E \otimes_{\pi^*(\mathcal{A}^{op})} G) \otimes G^\vee) \cong \pi_* \pi^* E \cong E$$

again by (1.6). Now for $E \in Coh(Y, X)$ we get

$$F(H(E)) = \pi^*(\pi_*(E \otimes G^\vee)) \otimes_{\pi^*(\mathcal{A}^{op})} G \cong (E \otimes G^\vee) \otimes_{\pi^*(\mathcal{A}^{op})} G \cong E$$

The first isomorphism follows from $E \in Coh(Y, X)$ and the second isomorphism follows from $G^\vee \otimes_{\pi^*(\mathcal{A}^{op})} G \cong \mathcal{O}_Y$, see ([GW10, Proposition 8.26.]).

This shows that these two categories are equivalent and therefore $Coh_l(X, \mathcal{A})$ and $Coh(Y, X)$ are also equivalent. \square

2 Moduli spaces of line bundles on a noncommutative Azumaya surface

In this section we work again with a fixed scheme X , which in this case should be a smooth projective $K3$ -surface over \mathbb{C} . Thus we have the Mukai pairing $\langle -, - \rangle$ on $H^{2*}(X, \mathbb{Q})$ given by

$$\langle x, y \rangle = - \int_X x^\vee y \quad x, y \in H^{2*}(X, \mathbb{Q}),$$

see for example ([HL10, 2.6.1.5]). To shorten notation we write x^2 for the term $\langle x, x \rangle$. We want to study moduli spaces of sheaves on X , so we need to understand families of sheaves. For this we pick an Azumaya algebra \mathcal{A} of rank r^2 on X and let $\pi : Y \rightarrow X$ be the associated Brauer-Severi variety.

For every $S \in Sch_{\mathbb{C}}$ and for every $s \in S$ we have the double pullback diagram:

$$\begin{array}{ccccc} Y_s & \xrightarrow{j_s} & Y \times S & \xrightarrow{q} & Y \\ \pi_s \downarrow & & \pi_S \downarrow & & \downarrow \pi \\ X_s & \xrightarrow{i_s} & X \times S & \xrightarrow{p} & X \end{array}$$

where we have $Y \times S \cong BS(\mathcal{A}_S)$ and $Y_s \cong BS(\mathcal{A}_s)$ by (1.4) so that both schemes are also Brauer-Severi varieties.

The following objects live on the various schemes in this diagram:

- \mathcal{A} on X , $\mathcal{A}_S := p^* \mathcal{A}$ on $X \times S$ and $\mathcal{A}_s := i_s^* p^* \mathcal{A}$ on X_s .
- $\pi^*(\mathcal{A}^{op}) \cong \mathcal{E}nd_{\mathcal{O}_Y}(G)$ on Y , $\pi_S^*(\mathcal{A}_S^{op}) \cong \mathcal{E}nd_{\mathcal{O}_{Y \times S}}(G_S)$ on $Y \times S$ with $G_S = q^* G$ and $\pi_s^*(\mathcal{A}_s^{op}) \cong \mathcal{E}nd_{\mathcal{O}_{Y_s}}(G_s)$ on Y_s with $G_s = j_s^* q^* G$.

Remark 2.1. At the generic point $\eta \in X$ the Azumaya algebra \mathcal{A} is given by the central simple algebra $\mathcal{A}_\eta = M_n(D)$ for some division ring D over the function field $\mathbb{C}(X)$. Without loss of generality we may assume $n = 1$. This is because we are only interested in the Brauer class $[\mathcal{A}] \in Br(X)$. (Brauer equivalent algebras have equivalent module categories and hence isomorphic moduli spaces.) But we have $[M_n(D)] = [D] \in Br(\mathbb{C}(X))$ so by injectivity of $Br(X) \rightarrow Br(\mathbb{C}(X))$, $[\mathcal{A}] \mapsto [\mathcal{A}_\eta]$ all we need to do is find an Azumaya algebra \mathcal{D} on X with $\mathcal{D}_\eta = D$. The existence of such an algebra is due the following theorem:

Theorem 2.2. *Assume X is a regular integral scheme of dimension at most two with function field K . If A is a central simple K -algebra of dimension n^2 whose image in ${}_n Br(K)$ is unramified at every point of codimension one in X , then there is an Azumaya algebra \mathcal{A} of rank n^2 on X such that $\mathcal{A} \otimes K = A$.*

Proof. See ([CT04, Théorème 2.5.]). □

Using this we see that a generically simple \mathcal{A} -module E , i.e. E_η is a simple \mathcal{A}_η -module, must be generically of rank one over \mathcal{A} , hence has rank r^2 over X . We call such modules \mathcal{A} -modules of rank one. Now we can study the moduli functors of interest.

First we define

$$\mathcal{M}_{\mathcal{A}/X}(v_{\mathcal{A}}) : (Sch_{\mathbb{C}})^{op} \rightarrow Sets$$

which sends a \mathbb{C} -scheme S to the set of isomorphism classes of families of torsion-free \mathcal{A} -modules of rank one with Mukai vector $v_{\mathcal{A}}$ over S .

The following facts are known:

Theorem 2.3. *The moduli functor $\mathcal{M}_{\mathcal{A}/X}(v_{\mathcal{A}})$ has a coarse moduli scheme $M_{\mathcal{A}/X}(v_{\mathcal{A}})$. Furthermore $M_{\mathcal{A}/X}(v_{\mathcal{A}})$ is a smooth projective scheme with a symplectic form on its tangent bundle.*

Proof. See ([HS05, Theorem 2.4., Theorem 3.6.]). □

The other moduli functor of interest is the following:

$$\mathcal{M}_H^{Y,G}(v_G) : (Sch_{\mathbb{C}})^{op} \rightarrow Sets$$

which maps a \mathbb{C} -scheme S to the set of isomorphism classes of families of torsion-free G -twisted semistable \mathcal{O}_Y -modules of rank r with Mukai vector v_G over S .

This moduli functor has also been well studied:

Theorem 2.4. *The moduli functor $\mathcal{M}_H^{Y,G}(v_G)$ has a coarse moduli scheme $M_H^{Y,G}(v_G)$. If v_G is primitive and H is general with respect to v_G , then all G -twisted semistable sheaves are G -twisted stable. In particular $M_H^{Y,G}(v_G)$ is a smooth projective scheme with a symplectic form on its tangent bundle.*

Proof. See ([Yos06, Theorem 2.1., Proposition 3.6., Theorem 3.11.]). □

Remark 2.5. By definition, ([Ree13, Definition 2.7]), we have for $E \in \text{Coh}_l(X, \mathcal{A})$:

$$v_{\mathcal{A}}(E) = \frac{ch(E)}{\sqrt{ch(\mathcal{A})}} \sqrt{td(X)}.$$

Furthermore in ([Yos06, Definition 3.1.]) Yoshioka defined for $E \in \text{Coh}(Y, X)$:

$$v_G(E) = \frac{ch(\pi_*(E \otimes G^\vee))}{\sqrt{ch(\pi_*(G \otimes G^\vee))}} \sqrt{td(X)}.$$

(Actually Yoshioka defines this vector using the derived direct image, but the sheaf $E \otimes G^\vee$ does not have higher direct images for $E \in \text{Coh}(Y, X)$ by (1.6).)

Using the equivalence from (1.9) we have $v_G(F(E)) = v_{\mathcal{A}}(E)$ and $v_{\mathcal{A}}(H(E)) = v_G(E)$. This is because $\mathcal{A}^{op} \cong \pi_*(G \otimes G^\vee)$ and $ch(\mathcal{A}^{op}) = ch(\mathcal{A})$. So these Mukai vectors are the same and in the following we will omit the subscript and just write v for a fixed Mukai vector.

If $E \in \text{Coh}_l(X, \mathcal{A})$ is given such that E is a torsion-free \mathcal{A} -module of rank one with Mukai vector $v = v_{\mathcal{A}}(E)$, then we have $rk(E) = r^2$. Now E has no \mathcal{A} -submodules $E' \subsetneq E$ with $0 < rk(E') < r^2$ because any such module must satisfy $r^2 | rk(E')$ which is impossible. So $F(E)$ is a torsion-free \mathcal{O}_Y -module of rank r , has Mukai vector $v_G = v$ and has no submodules in $\text{Coh}(Y, X)$ since F preserves submodules. $F(E)$ is therefore G -twisted stable.

On the other hand if $E \in \text{Coh}(Y, X)$ is given and E is a torsion-free G -twisted semistable \mathcal{O}_Y -module of rank r with $v_G(E) = v$ then $H(E) \in \text{Coh}_l(X, \mathcal{A})$ and $H(E)$ is torsion-free and of rank one, as $rk(H(E)) = r^2$. We can conclude that E must be G -twisted stable, because any submodule $E' \subsetneq E$ in $\text{Coh}(Y, X)$ with $0 < rk(E') < r$ would give rise to an \mathcal{A} -submodule $H(E') \subsetneq H(E)$ in $\text{Coh}_l(X, \mathcal{A})$ with $0 < rk(H(E')) < r^2$ as H also preserves submodules. But this is impossible since $H(E)$ is a torsion-free \mathcal{A} -module of rank one.

We have thus proven the following lemma:

Lemma 2.6. *Assume v is a Mukai vector with $v = v_{\mathcal{A}}(E)$ for some torsion-free \mathcal{A} -module E of rank one, then any torsion-free G -twisted semistable \mathcal{O}_Y -module of rank r with Mukai vector $v_G = v$ is G -twisted stable. Under the equivalence $\text{Coh}_l(X, \mathcal{A}) \cong \text{Coh}(Y, X)$ torsion-free \mathcal{A} -modules of rank one with Mukai vector v correspond to torsion-free G -twisted stable \mathcal{O}_Y -modules of rank r with Mukai vector v .*

Remark 2.7. The previous lemma (2.6) shows that given a Mukai vector $v = v_{\mathcal{A}}(E)$ for some torsion-free \mathcal{A} -module E , we do not need to worry if the polarization H is general with respect to v , see (2.4). All torsion-free G -twisted semistable \mathcal{O}_Y -modules of rank r and with Mukai vector $v_G = v$ are automatically G -twisted stable. So $M_H^{Y,G}(v)$ is a smooth projective scheme without choosing a special polarization H .

Looking at the definition of the moduli functors, we see that we are working with two kind of families of sheaves:

- i) a family of torsion-free \mathcal{A} -modules of rank one with Mukai vector v is a sheaf $\mathcal{E} \in \text{Coh}_l(X \times S, \mathcal{A}_S)$ such that \mathcal{E} is flat over S and $\mathcal{E}_s = i_s^* \mathcal{E}$ is a torsion-free \mathcal{A}_s -module of rank one on X_s with Mukai vector v for every closed point $s \in S$, especially $\mathcal{E}_s \in \text{Coh}_l(X_s, \mathcal{A}_s)$.
- ii) a family of torsion-free G -twisted semistable \mathcal{O}_Y -modules of rank r with Mukai vector v is a sheaf $\mathcal{F} \in \text{Coh}(Y \times S, X \times S)$ such that \mathcal{F} is flat over S and $\mathcal{F}_s = j_s^* \mathcal{F}$ is a torsion-free G -twisted semistable \mathcal{O}_Y -module of rank r on Y_s with Mukai vector v for every closed point $s \in S$, especially $\mathcal{F}_s \in \text{Coh}(Y_s, X_s)$.

Now for every $S \in \text{Sch}_{\mathbb{C}}$ we have $X \times S \in \text{Sch}_{\mathbb{C}}$. Furthermore \mathcal{A}_S is an Azumaya algebra on $X \times S$ with a canonical isomorphism $B\mathcal{S}(\mathcal{A}_S) \cong Y \times S$ so lemma (1.9) gives us an equivalence of categories

$$\text{Coh}_l(X \times S, \mathcal{A}_S) \cong \text{Coh}(Y \times S, X \times S)$$

which has the following property:

Lemma 2.8. *The equivalence $\text{Coh}_l(X \times S, \mathcal{A}_S) \cong \text{Coh}(Y \times S, X \times S)$ maps families of type i) to families of type ii) with semistable replaced by stable and vice versa.*

Proof. Let \mathcal{E} be a family of type i). Define $\mathcal{F} := \pi_S^* \mathcal{E} \otimes_{\pi_S^*(\mathcal{A}_S^{op})} G_S$, then \mathcal{F} is a family of type ii). We have $\mathcal{F} \in \text{Coh}(Y \times S, X \times S)$ and \mathcal{F} is flat over S . To see this note that π_S is faithfully flat, so $\pi_S^* \mathcal{E}$ is flat over S , see ([GD65, 2.2.11 (iii)]). Furthermore G_S is a flat $\pi_S^*(\mathcal{A}_S^{op})$ -module, so \mathcal{F} is flat over S . We also see that

$$\mathcal{F}_s = j_s^*(\pi_S^* \mathcal{E} \otimes_{\pi_S^*(\mathcal{A}_S^{op})} G_S) = \pi_s^* \mathcal{E}_s \otimes_{\pi_s^*(\mathcal{A}_s^{op})} G_s.$$

So $\mathcal{F}_s \in \text{Coh}(Y_s, X_s)$ and \mathcal{F}_s is a torsion-free G -twisted stable \mathcal{O}_Y -module of rank r on Y_s with Mukai vector v by (2.6).

Let \mathcal{F} be a family of type ii). Define $\mathcal{E} := \pi_{S*}(\mathcal{F} \otimes G_S^\vee)$, then \mathcal{E} is family of type i). We have $\mathcal{E} \in \text{Coh}_l(X \times S, \mathcal{A}_S)$ and \mathcal{E} is flat over S . To see this we note that this can be tested after pullback with a faithfully flat morphism $f : Z \rightarrow X \times S$ by ([GD65, 2.2.11 (iii)]), so we just use $f = \pi_S$. But $\pi_S^* \mathcal{E} = \pi_S^* \pi_{S*}(\mathcal{F} \otimes G_S^\vee) \cong \mathcal{F} \otimes G_S^\vee$ and the latter is flat over S since \mathcal{F} and G_S^\vee are. Finally

$$\begin{aligned} \mathcal{E}_s &= i_s^* \pi_{S*}(\mathcal{F} \otimes G_S^\vee) \cong \pi_{s*} \pi_s^* i_s^* \pi_{S*}(\mathcal{F} \otimes G_S^\vee) \\ &= \pi_{s*} j_s^* \pi_S^* \pi_{S*}(\mathcal{F} \otimes G_S^\vee) \cong \pi_{s*} j_s^*(\mathcal{F} \otimes G_S^\vee) = \pi_{s*}(\mathcal{F}_s \otimes G_s^\vee). \end{aligned}$$

So $\mathcal{E}_s \in \text{Coh}_l(X_s, \mathcal{A}_s)$ and \mathcal{E}_s is a torsion-free \mathcal{A}_s -module of rank one on X_s with Mukai vector v by (2.6). \square

Theorem 2.9. *Assume $v = v_{\mathcal{A}}(E)$ is a Mukai vector for some torsion-free \mathcal{A} -module E of rank one, then the functors $\mathcal{M}_{\mathcal{A}/X}(v)$ and $\mathcal{M}_H^{Y,G}(v)$ are isomorphic.*

Proof. We define a natural transformation $\eta : \mathcal{M}_{\mathcal{A}/X}(v) \rightarrow \mathcal{M}_H^{Y,G}(v)$:

For $S \in \text{Sch}_{\mathbb{C}}$ we let $\eta_S : \mathcal{M}_{\mathcal{A}/X}(v)(S) \rightarrow \mathcal{M}_H^{Y,G}(v)(S)$ be the map

$$[\mathcal{E}] \mapsto [\pi_S^* \mathcal{E} \otimes_{\pi_S^*(\mathcal{A}_S^{op})} G_S].$$

This map is well-defined by (2.8). Let $T \in \text{Sch}_{\mathbb{C}}$ with $f : T \rightarrow S$. Then we have

$$\begin{aligned} \mathcal{M}_{\mathcal{A}/X}(v)(f) : \mathcal{M}_{\mathcal{A}/X}(v)(S) &\rightarrow \mathcal{M}_{\mathcal{A}/X}(v)(T), \quad [\mathcal{E}] \mapsto [f_X^*(\mathcal{E})] \\ \mathcal{M}_H^{Y,G}(v)(f) : \mathcal{M}_H^{Y,G}(v)(S) &\rightarrow \mathcal{M}_H^{Y,G}(v)(T), \quad [\mathcal{F}] \mapsto [f_Y^*(\mathcal{F})] \end{aligned}$$

where $f_X = id_X \times f$ and $f_Y = id_Y \times f$. This gives the diagram:

$$\begin{array}{ccccc} Y \times T & \xrightarrow{f_Y} & Y \times S & \xrightarrow{q} & Y \\ \pi_T \downarrow & & \pi_S \downarrow & & \downarrow \pi \\ X \times T & \xrightarrow{f_X} & X \times S & \xrightarrow{p} & X \end{array}$$

One computes:

$$\begin{aligned}\eta_T(\mathcal{M}_{\mathcal{A}/X}(v)(f)[\mathcal{E}]) &= [\pi_T^* f_X^* \mathcal{E} \otimes_{\pi_T^*(\mathcal{A}_T^{op})} G_T] \\ &= [f_Y^* \pi_S^* \mathcal{E} \otimes_{f_Y^*(\pi_S^*(\mathcal{A}_S^{op}))} f_Y^* G_S] = \mathcal{M}_H^{Y,G}(v)(f)(\eta_S([\mathcal{E}]))\end{aligned}$$

Hence

$$\eta_T \circ \mathcal{M}_{\mathcal{A}/X}(v)(f) = \mathcal{M}_H^{Y,G}(v)(f) \circ \eta_S.$$

So η is in fact a natural transformation.

Define a second natural transformation $\Psi : \mathcal{M}_H^{Y,G}(v) \rightarrow \mathcal{M}_{\mathcal{A}/X}(v)$ by

$$\Psi_S : \mathcal{M}_H^{Y,G}(v)(S) \rightarrow \mathcal{M}_{\mathcal{A}/X}(v)(S), \quad [\mathcal{F}] \mapsto [\pi_{S*}(\mathcal{F} \otimes G_S^\vee)].$$

Again this map is well-defined by (2.8). Then one sees

$$\begin{aligned}\Psi_T(\mathcal{M}_H^{Y,G}(v)(f)[\mathcal{F}]) &= [\pi_{T*}(f_Y^* \mathcal{F} \otimes G_T^\vee)] = [\pi_{T*}(f_Y^*(\mathcal{F} \otimes G_S^\vee))] \\ &= [\pi_{T*}(f_Y^*(\pi_S^* \pi_{S*}(\mathcal{F} \otimes G_S^\vee)))] \\ &= [\pi_{T*} \pi_T^*(f_X^*(\pi_{S*}(\mathcal{F} \otimes G_S^\vee)))] \\ &= [f_X^*(\pi_{S*}(\mathcal{F} \otimes G_S^\vee))] = \mathcal{M}_{\mathcal{A}/X}(v)(f)(\Psi_S([\mathcal{F}]))\end{aligned}$$

and hence $\Psi_T \circ \mathcal{M}_H^{Y,G}(v)(f) = \mathcal{M}_{\mathcal{A}/X}(v)(f) \circ \Psi_S$.

Finally by what we have already seen η_S and Ψ_S are inverse bijections for every S , so η is a natural isomorphism between these moduli functors. \square

Corollary 2.10. *Assume $v = v_{\mathcal{A}}(E)$ is a Mukai vector for some torsion-free \mathcal{A} -module E of rank one, then the moduli spaces $M_{\mathcal{A}/X}(v)$ and $M_H^{Y,G}(v)$ are isomorphic.*

Proof. Since these schemes are coarse moduli spaces they corepresent their corresponding moduli functors, see ([HL10, Definition 2.2.1]). The natural isomorphisms η and Ψ thus induce natural isomorphisms $Hom_{Sch_{\mathbb{C}}}(-, M_{\mathcal{A}/X}(v)) \cong Hom_{Sch_{\mathbb{C}}}(-, M_H^{Y,G}(v))$ by the universal property. But then Yoneda implies $M_{\mathcal{A}/X}(v) \cong M_H^{Y,G}(v)$. \square

Corollary 2.11. *Assume $v = v_{\mathcal{A}}(E)$ is a primitive Mukai vector for some torsion-free \mathcal{A} -module E of rank one, then $M_{\mathcal{A}/X}(v)$ is an irreducible symplectic variety deformation equivalent to $Hilb^{\frac{n}{2}+1}(X)$. Moreover we have:*

- $M_{\mathcal{A}/X}(v) \neq \emptyset$ if and only if $v^2 \geq -2$
- if $v^2 = 0$, then $M_{\mathcal{A}/X}(v)$ is a K3-surface.

Proof. Using (2.10) we have $M_{\mathcal{A}/X}(v) \cong M_H^{Y,G}(v)$. Now the statements follow from ([Yos06, Theorem 3.16.]). \square

Corollary 2.12. *Assume $v = v_{\mathcal{A}}(E)$ is a primitive Mukai vector for some torsion-free \mathcal{A} -module E of rank one, then we have:*

- $h^{p,q}(M_{\mathcal{A}/X}(v)) = h^{p,q}(Hilb^{\frac{n}{2}+1}(X))$
- $b_i(M_{\mathcal{A}/X}(v)) = b_i(Hilb^{\frac{n}{2}+1}(X))$

Here $h^{p,q}$ are the Hodge numbers and b_i are the Betti numbers.

Proof. This follows from (2.11) using the fact that the Betti numbers and the Hodge numbers are invariant with respect deformation equivalence. \square

References

- [CT04] J.-L. Colliot-Thélène. Algèbres simples centrales sur les corps de fonctions de deux variables. *Séminaire Bourbaki*, 47:379–414, 2004.
- [GD65] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Second partie*. Publications Mathématiques de l’IHÉS (24), 1965.
- [Gro68] A. Grothendieck. Le groupe de Brauer: I. Algèbres d’Azumaya et interprétations diverses. In *Dix Exposés sur la Cohomologie des Schémas*, pages 46–66. North-Holland, 1968.
- [GW10] U. Görtz and T. Wedhorn. *Algebraic Geometry I*. Vieweg+Teubner, 2010.
- [HL10] D. Huybrechts and M. Lehn. *The Geometry of the Moduli Spaces of Sheaves*. Cambridge University Press, Second edition, 2010.
- [HS05] N. Hoffmann and U. Stuhler. Moduli schemes of generically simple Azumaya modules. *Documenta Mathematica*, 10:369–389, 2005.
- [Qui73] D. Quillen. Higher algebraic K-theory I. *Higher K-Theories (Lecture Notes in Mathematics 341)*, pages 85–147, 1973.
- [Ree13] F. Reede. *Moduli spaces of bundles over two-dimensional orders*. PhD thesis, Mathematisches Institut der Georg-August-Universität, Göttingen, 2013. URL: <http://hdl.handle.net/11858/00-1735-0000-001A-7778-7>.
- [TT90] R.W. Thomasson and T. Trobaugh. Higher Algebraic K-Theory of Schemes and of Derived Categories. In *The Grothendieck Festschrift, Volume III*, pages 247–435. Birkhäuser, 1990.
- [Yos06] K. Yoshioka. Moduli spaces of twisted sheaves on a projective variety. *Advanced Studies in Pure Mathematics*, 45:1–42, 2006.